

Particle creation from the vacuum by an exponentially decreasing electric field

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Abstract

We analyze the creation of fermions and bosons from the vacuum by the exponentially decreasing in time electric field in detail. In our calculations we use QED and follow in main the consideration of particle creation effect in a homogeneous electric field. To this end we find complete sets of exact solutions of the d -dimensional Dirac equation in the exponentially decreasing electric field and use them to calculate all the characteristics of the effect, in particular, the total number of created particles and the probability of a vacuum to remain a vacuum. It should be noted that the latter quantities were derived in the case under consideration for the first time. All possible asymptotic regimes are discussed in detail. In addition, switching on and switching off effects are studied.

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I. INTRODUCTION

Particle creation from the vacuum by strong external electromagnetic fields is an important nonperturbative effect, theoretical study of which has a long history, see for example the Refs. [1–7]. To be observable, the effect needs very strong electric fields in magnitudes compared with the Schwinger critical field $E_c = m^2 c^3 / e \hbar \simeq 1.3 \times 10^{16} \text{ V} \cdot \text{cm}^{-1}$. However, recent progress in laser physics allows one to hope that the nonperturbative regime of pair production may be reached in the near future, see Ref. [8] for the review. Electron-hole pair creation from the vacuum becomes also an observable in the laboratory effect in graphene physics, an area that is currently under intense development [9, 10]. In particular, this effect is crucial for understanding the conductivity of the graphene, especially in the so-called nonlinear regime, see, for example, Ref. [11]. The particle creation from the vacuum by external electric and gravitational backgrounds plays also an important role in cosmology and astrophysics [6].

It should be noted that the particle creation from the vacuum by external fields is a nonperturbative effect and its calculation essentially depends on the structure of the external fields. Sometimes calculations can be done in the framework of the relativistic quantum mechanics, sometimes using semiclassical and numerical methods (see Refs. [6, 8, 12] for the review). The vast majority of analytic works in this area, in QED, is based on the worldline and instanton formalisms, rather than solving the Dirac equation (for example, see [13, 14] and references therein). In all these cases the authors calculate, in fact, the one-loop effective action, whose imaginary part is related to the probability of a vacuum to remain a vacuum. However, in those cases, when the semiclassical approximation does not work, the most convinced consideration is formulated in the framework of QFT, in particular, in the framework of QED, see Ref. [3, 4, 7]. In particular, in the latter approach nonperturbative calculations are based on the existence of exact solutions of the Dirac equation with the corresponding external electromagnetic field. In fact, until now, there are known only few exactly solvable cases for either time-dependent homogeneous or constant inhomogeneous electric fields. One of them is related to the constant uniform electric field [1], another one to the so-called adiabatic electric field $E(t) = E \cosh^{-2}(t/\alpha)$ [15] (see also [16]), the case related to the so-called T -constant electric field [17–19], which corresponds to a constant electric field that turns-on and -off at definite times instants t_1 and t_2 , ($t_2 - t_1 = T$) being

constant inside of the time interval T , the case related to a periodic alternating electric field [20], and the number of a constant inhomogeneous electric fields of the similar forms where time t is replaced by the spatial coordinate x . To complete the picture, we note that these exist exact solutions of the Dirac equation with some electric fields satisfying more complicated symmetries, e.g. with potentials given in the light-cone variables, for example, see [21] and [22]. The existence of exactly solvable cases of particle creation is extremely important both for deep understanding of QFT in general and for studying quantum vacuum effects in the corresponding external fields.

In this article, we present a new exactly solvable case of particle creation that corresponds to the so-called T -exponentially decreasing in time electric field, which switches on at the time instant t_1 , switches off at the time instant t_2 ($t_2 - t_1 = T$), and within the time interval T has the form $E_x(t) = Ee^{-k_0(t-t_1)}$, where k_0, E are some positive constants. In particular, this field presents the example of an exponentially decaying electric field when $t_2 \rightarrow \infty$. Technically this exactly solvable case differs essentially from all the above mentioned cases because of an asymmetrical asymptotic behavior of the external electric field. Consideration of such a case has an interesting physical motivation. The corresponding external electric field can be treated as one, which is created by an external current that switches on fast enough and then is slowly switching off (decreases) because of some dissipation processes. One can demonstrate that under certain conditions the main contribution to particle creation is due to the decreasing part of the electric field, whereas the contribution from the increasing part of the field is relatively small. The qualitative difference in the asymptotic behavior of the external electric field under consideration allows one to study the role of switching on and switching off for an electric field. We just from the beginning consider general ($d = D + 1$)-dimensional Minkowski space-time, to be able to use the case $D = 3$ for describing high-energy effects, while the case $D = 1, 2, 3$ could be adequate for condense matter problems. For completeness, the case of scalar particles is considered too.

It is worth to note that the differential mean number of particles created by a kind of exponentially decaying electric field was calculated previously in the Ref. [23] in the framework of some semiclassical considerations and in [24] using the Dirac-Heisenberg-Wigner function. However, the authors of the latter work did not present any analysis how their results depend on the problem parameters in the case of a strong field, in fact, they studied the weak field limit only.

As was already said, in our calculations, we use the general theory of Ref. [3, 4] and follow in main the consideration of particle creation effect in a homogeneous electric field [18], see appendix A for some basic elements. To this end we find complete sets of exact solutions of the Dirac and Klein-Gordon equations in the T -exponentially decreasing electric field and use them to calculate all the characteristics of the effect, in particular, the differential mean number of particle created, total number of created particles, and the probability for a vacuum to remain a vacuum. It should be noted that the latter quantities were derived in the case under consideration for the first time. Using these solutions, we analyze particle creation in the case of the exponentially decaying electric field. All possible asymptotic regimes are discussed in detail. In addition, switching on and switching off effects are studied.

II. EXPONENTIALLY DECREASING ELECTRIC FIELD

We consider the Dirac equation¹ in $(d = D + 1)$ -dimensional Minkowski space with an external electromagnetic field given by potentials $A_\mu(x)$,

$$\left(\gamma^\mu \hat{P}_\mu - m\right) \psi(x) = 0, \quad \hat{P}_\mu = \hat{p}_\mu - qA_\mu(x), \quad \hat{p}_\mu = i\partial_\mu. \quad (2.1)$$

Here $\psi(x)$ is a $2^{[d/2]}$ -component spinor ($[d/2]$ stands for the integer part of $d/2$), m is the particle mass, q is the particle charge (for the electron $q = -e$, with $e > 0$ being the absolute value of the electron charge), $x = (x^\mu) = (x^0, \mathbf{x})$, $\mathbf{x} = (x^i)$, $x^0 = t$, the Greek and Latin indexes assume values $\mu = 0, 1, \dots, D$ and $i = 1, \dots, D$ respectively, and γ -matrices satisfy the standard anticommutation relations:

$$[\gamma^\mu, \gamma^\nu]_+ = 2\eta^{\mu\nu}, \quad \eta_{\mu\nu} = \text{diag}(1, -1, \dots, -1).$$

Using the Ansatz $\psi(x) = \left(\gamma^\mu \hat{P}_\mu + m\right) \phi(x)$, one finds that the spinor $\phi(x)$ satisfies the following equation:

$$\begin{aligned} \left(\hat{P}^2 - m^2 - \frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu}\right) \phi(x) &= 0, \\ \sigma^{\mu\nu} &= \frac{i}{2}[\gamma^\mu, \gamma^\nu], \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \quad (2.2)$$

¹ From this section and in what follows we are considering the system of units $\hbar = c = 1$ and the fine structure constant is $\alpha = e^2$.

In what follows, we consider the so-called T -exponentially decreasing electric field with a constant direction along the x axis. This field switches on at t_1 and switches off at t_2 , being nonzero within the time interval $T = t_2 - t_1 > 0$ and zero outside of it,

$$E_x(t) = E \begin{cases} 0, & t \in \text{I} = (-\infty, t_1) \\ e^{-k_0(t-t_1)}, & t \in \text{II} = [t_1, t_2] \\ 0, & t \in \text{III} = (t_2, +\infty) \end{cases}, \quad k_0 > 0. \quad (2.3)$$

We choose the corresponding potentials as $A^\mu(t) = \delta_1^\mu A_x(t)$ with only one nonzero component,

$$A_x(t) = \frac{E}{k_0} \begin{cases} 1, & t \in \text{I} \\ e^{-k_0(t-t_1)}, & t \in \text{II} \\ e^{-k_0 T}, & t \in \text{III} \end{cases}. \quad (2.4)$$

We admit that the switching off can occur in the remote future such that t_2 can be infinite, $t_2 = +\infty$, under the condition that t_1 remains finite.

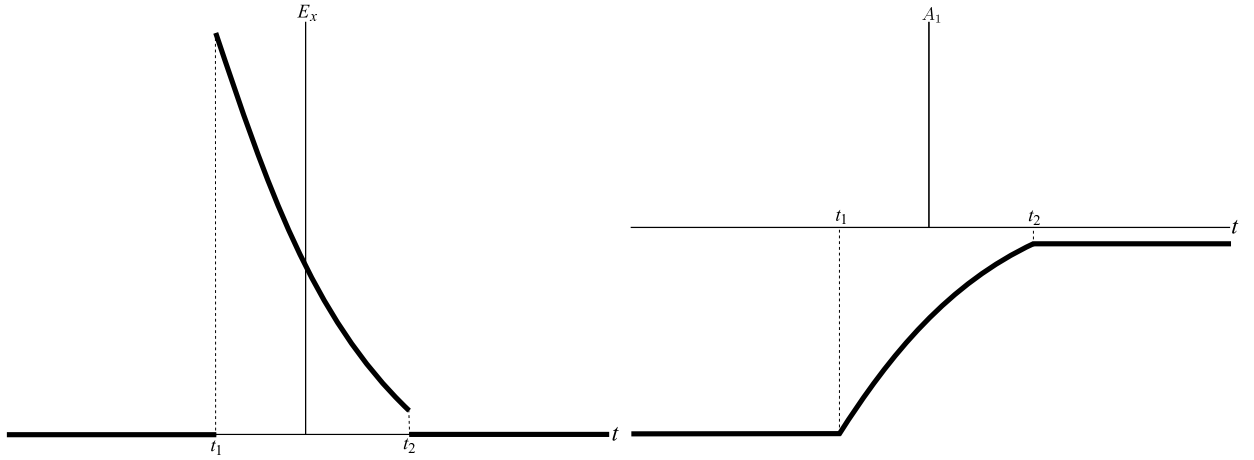


FIG. 1. The exponentially decreasing electric field and its potential.

Solving equation (2.2), we will use a set of constant orthonormalized spinors $v_{s,\sigma}$,

$$v_{s,\sigma}^\dagger v_{s',\sigma'} = \delta_{s,s'} \delta_{\sigma,\sigma'}, \quad v v^\dagger = \mathbb{I},$$

with $s = \pm 1$, and $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{[d/2]-1})$, $\sigma_j = \pm 1$, such that $\gamma^0 \gamma^1 v_{s,\sigma} = s v_{s,\sigma}$. For the $d \geq 4$ the indices σ_j describe the spin polarization, which is not coupled to the electric field, and together with the additional index s provide a suitable parametrization of the solutions. Note that for the $d = 2, 3$ there is only one spin degree of freedom and the spinors are labeled

either by $s = +1$ or by $s = -1$. Solutions of eq. (2.2) with the potential given by eq. (2.4) can be found in the form

$$\phi_{\mathbf{p},s,\sigma}(x) = \varphi_{\mathbf{p},s}(t)e^{i\mathbf{p}\cdot\mathbf{x}}v_{s,\sigma}, \quad (2.5)$$

where scalar functions $\varphi_{\mathbf{p},s}(t)$ satisfy the following second order differential equation

$$\left[\frac{d^2}{dt^2} + [p_x - qA_x(t)]^2 + \mathbf{p}_\perp^2 + m^2 + isqE_x(t) \right] \varphi_{\mathbf{p},s}(t) = 0, \quad (2.6)$$

where \mathbf{p}_\perp is the transversal particle momentum, $\mathbf{p}_\perp = (0, p^2, \dots, p^D)$.

Thus, in what follows, we are going to deal with two complete set of solutions of the Dirac equation (2.1) of the following structure

$$\begin{aligned} \pm\psi_n(x) &= \left(\gamma^\mu \hat{P}_\mu + m \right) \pm\phi_{\mathbf{p},s,\sigma}(x), \\ \pm\psi_n(x) &= \left(\gamma^\mu \hat{P}_\mu + m \right)^\pm \phi_{\mathbf{p},s,\sigma}(x), \end{aligned} \quad (2.7)$$

where spinors $\pm\phi_{\mathbf{p},s,\sigma}(x)$ and ${}^\pm\phi_{\mathbf{p},s,\sigma}(x)$ are given by Eq. (2.5) with solutions ${}_\zeta\varphi_{\mathbf{p},s}(t)$ and ${}^\zeta\varphi_{\mathbf{p},s}(t)$, respectively, satisfying Eq. (2.6) with initial or final conditions that are specified in which follows. Here we denote by $n = (\mathbf{p}, \sigma)$ a complete set of quantum numbers of the Dirac spinor for both cases $s = +1$ or $s = -1$. Note that for the $d \geq 4$ the Dirac spinors given by the choice of $s = +1$ in (2.7) are linearly dependent with the spinors given by the choice of $s = -1$. Thus, one can form physically equivalent complete sets of the Dirac spinors for both choices of parametrization. The algebra of γ -matrices has two inequivalent representations in $d = 3$ dimensions, representation given by $s = +1$ and $s = -1$ are associated with different fermion species.

Note that a formal reduction to the spinless case that corresponds to the use of the Klein-Gordon equation instead of the Dirac one can be done by setting $s = 0$ in (2.6) and $v_{s,\{r\}} = 1$ in (2.5). In this case $n = (\mathbf{p})$.

In the first region I and in the third region III the electric field is absent and Eq. (2.6) has plane wave solutions ${}_\zeta\varphi_{\mathbf{p},s}(t)$ and ${}^\zeta\varphi_{\mathbf{p},s}(t)$, respectively, with additional quantum number $\zeta = \pm$, which satisfy simple dispersion relations

$$\begin{aligned} \text{I : } {}_\zeta\varphi_{\mathbf{p},s}(t) &\sim e^{-i\zeta p_0(t_1)t}, \quad \text{III : } {}_\zeta\varphi_{\mathbf{p},s}(t) \sim e^{-i\zeta p_0(t_2)t}, \\ p_0(t) &= \sqrt{\left(p'_x - \frac{|qE|}{k_0} e^{-k_0(t-t_1)} \right)^2 + \mathbf{p}_\perp^2 + m^2}, \end{aligned} \quad (2.8)$$

where $p'_x = \varkappa p_x$ and $\varkappa = \text{sgn}(qE)$. Here, the quantum numbers ζ label particle/antiparticles states such that positive ($\zeta = +$)/negative ($\zeta = -$) values define particles/antiparticles states, respectively.

In the second region II, it is convenient to introduce a new variable η and represent the functions $\varphi_{\mathbf{p},s}$ as

$$\begin{aligned} \eta &= \frac{2i|qE|}{k_0^2} e^{-k_0(t-t_1)}, \quad \varphi_{\mathbf{p},s}(t) = e^{-\eta/2} \eta^\nu \chi_{\mathbf{p},s}(\eta), \\ \nu &= i \frac{\omega_0}{k_0}, \quad \omega_0 = \sqrt{\mathbf{p}^2 + m^2}. \end{aligned} \quad (2.9)$$

Then the functions $\chi_{\mathbf{p},s}(\eta)$ satisfy the confluent hypergeometric equation [25],

$$\left[\eta \frac{d^2}{d\eta^2} + (c - \eta) \frac{d}{d\eta} - a \right] \chi_{\mathbf{p},s}(\eta) = 0,$$

with parameters

$$c = 1 + 2\nu, \quad a = \frac{1}{2} (1 - \varkappa s) - \frac{ip'_x}{k_0} + \nu. \quad (2.10)$$

The complete set of solutions for this equation is formed by two linearly independent confluent hypergeometric functions:

$$\Phi(a, c; \eta) \quad \text{and} \quad \eta^{1-c} \Phi(a - c + 1, 2 - c; \eta),$$

where

$$\Phi(a, c; \eta) = 1 + \frac{a}{c} \frac{\eta}{1!} + \frac{a(a+1)}{c(c+1)} \frac{\eta^2}{2!} + \dots$$

Thus one can find the general solution of the equation (2.6) in the time region II as the following linear superposition

$$\begin{aligned} \varphi(t) &= a_2 \varphi_1(t) + a_1 \varphi_2(t), \\ \varphi_1(t) &= e^{-\eta/2} \eta^\nu \Phi(a, c; \eta), \\ \varphi_2(t) &= e^{-\eta/2} \eta^{-\nu} \Phi(a - c + 1, 2 - c; \eta), \end{aligned} \quad (2.11)$$

where the constants a_1 and a_2 are fixed by initial conditions.

Taking into account expressions (2.8) and (2.11), one can construct orthonormalized solutions for the complete time interval in the following form

$$\zeta \varphi_{\mathbf{p},s}(t) = \begin{cases} g(-|\zeta) \quad -C e^{ip_0(t_1)(t-t_1)} + g(+|\zeta) \quad +C e^{-ip_0(t_1)(t-t_1)}, & t \in \text{I} \\ a_2^\zeta \varphi_1(t) + a_1^\zeta \varphi_2(t), & t \in \text{II} \\ \zeta C e^{-i\zeta p_0(t_2)(t-t_2)}, & t \in \text{III} \end{cases}, \quad (2.12)$$

where the constants ${}^\zeta C$ and ${}_\zeta C$ are defined by normalization conditions for the Dirac spinors (A4),

$$\begin{aligned} {}_\zeta C &= [2V p_0(t_1) p_\zeta(t_1)]^{-1/2}, \quad {}^\zeta C = [2V p_0(t_2) p_\zeta(t_2)]^{-1/2}, \\ p_\zeta(t) &= p_0(t) - \zeta s [p_x - q A_x(t)], \end{aligned} \quad (2.13)$$

and $p_0(t)$ is given by Eq. (2.8). Note that notation $g({}_\zeta|^\zeta)$ corresponds to definition (A6) from the Appendix. Coefficients a_1^ζ , a_2^ζ , $g(-|^\zeta)$, and $g(+|^\zeta)$ are specified by the following gluing conditions:

$${}^+ \varphi_{\mathbf{p},s}(t_k - 0) = {}^+ \varphi_{\mathbf{p},s}(t_k + 0), \quad \partial_t {}^+ \varphi_{\mathbf{p},s}(t)|_{t=t_k-0} = \partial_t {}^+ \varphi_{\mathbf{p},s}(t)|_{t=t_k+0}, \quad k = 1, 2. \quad (2.14)$$

It can be seen from the consideration given in the appendix A that the probability of a vacuum to remain a vacuum, the probability of a particle scattering, a pair creation, and a pair annihilation can be expressed via the differential mean number of particles created from vacuum N_n given by Eq.(A9). From which it follows that one can describe a vacuum instability for the case under consideration using the quantity

$$N_{\mathbf{p},\sigma} = |g(-|^\sigma)|^2 \quad (2.15)$$

only. Then it is enough to consider only the case $\zeta = +$ in Eq.(2.12). Using conditions (2.14), we obtain

$$a_1^+ = -\frac{i {}^+ C p_0(t_2)}{W} f_1(t_2), \quad a_2^+ = \frac{i {}^+ C p_0(t_2)}{W} f_2(t_2),$$

where W is the corresponding Wronskian of the solutions [25],

$$W = \varphi_1(t) \frac{d}{dt} \varphi_2(t) - \varphi_2(t) \frac{d}{dt} \varphi_1(t) = 2i\omega_0,$$

and

$$f_{1,2}(t) = \left[1 + \frac{ik_0\eta}{p_0(t)} \frac{d}{d\eta} \right] \varphi_{1,2}(t). \quad (2.16)$$

We finally find that the coefficient $g(-|^\sigma)$ takes the form

$$g(-|^\sigma) = \frac{1}{4\omega_0} \sqrt{\frac{p_0(t_2) p_-(t_1) p_0(t_1)}{p_+(t_2)}} [f_1(t_1) f_2(t_2) - f_2(t_1) f_1(t_2)]. \quad (2.17)$$

One can demonstrate that in the case of a sufficient long duration of the exponential electric field, when $t_2 \rightarrow +\infty$ and $p_0(t_1)(t_2 - t_1) \gg 1$, the differential mean numbers, given by

expression (2.17), coincide in the leading order term approximation with the result obtained in the Ref. [24].

Taking into account that the normalization constants ${}^\zeta C$ and ${}_\zeta C$ for the scalar case are

$${}^\zeta C = [2V p_0(t_1)]^{-1/2}, \quad {}_\zeta C = [2V p_0(t_2)]^{-1/2},$$

we find that in this case the coefficient $g(-|+)$ has the following form

$$g(-|+) = \frac{1}{4\omega_0} \sqrt{p_0(t_2) p_0(t_1)} [f_1(t_1) f_2(t_2) - f_2(t_1) f_1(t_2)], \quad (2.18)$$

where $f_{1,2}(t)$ are given by Eq. (2.16) at $s = 0$. The differential mean number of created scalar particles is expressed via $g(-|+)$ (2.18) as $N_{\mathbf{p}} = |g(-|+)|^2$.

Expression (2.17) does not depend on spin polarization parameters σ_j . That is why all the probabilities and the mean number do not depend on σ_j , so that the total (summed over all σ_j) probabilities and the mean number are $J_{(d)}$ times greater than the corresponding differential quantities. Here $J_{(d)} = 2^{[\frac{d}{2}]-1}$ is the number of spin degree of freedom. For example, the total number of particles created with a given momentum \mathbf{p} is

$$\sum_{\sigma} N_{\mathbf{p},\sigma} = J_{(d)} |g(-|+)|^2. \quad (2.19)$$

To get the total number N of created particles one has to sum over the spin projections, using eq.(2.19), and then over the momenta. The latter sum can be easily transformed into an integral,

$$N = \sum_{\mathbf{p}} \sum_{\sigma} N_{\mathbf{p},\sigma} = \frac{V J_{(d)}}{(2\pi)^{d-1}} \int d\mathbf{p} |g(-|+)|^2, \quad (2.20)$$

where V is $(d-1)$ -dimensional spatial volume.

The expression above depends essentially on the time interval of the field duration $T = t_2 - t_1$. Then the effect of pair creation depends on two dimensionless parameters $k_0 T$ and $|qE|/k_0^2$. For k_0 fixed, the first allows one to analyze the characteristics with respect to the time duration T of the electric field, while the second provides information on the maximum magnitude of the field $|E|$, for k_0 fixed too.

III. EXPONENTIALLY DECAYING STRONG FIELD

1. Differential quantities

Let us consider the exponentially decaying electric field given by Eq. (2.4), with

$$\frac{|qE|}{k_0^2} e^{-k_0 T} \ll 1, \quad (3.1)$$

when its initial magnitude is sufficiently large,

$$\frac{|qE|}{k_0^2} \gg K_f, \quad K_f \gg \max\left(\frac{\omega_0}{k_0}, 1\right), \quad (3.2)$$

where K_f is a given number. We stress that condition (3.2) corresponds to the most interesting case of a strong electric field where a perturbative consideration is not applicable.

In this case, using the asymptotics of the confluent hypergeometric functions [25], we first find from expression (2.17) that the differential mean numbers of created fermions are:

$$N_{\mathbf{p},\sigma} \simeq e^{-\frac{\pi}{k_0}(\omega_0 - p'_x)} \frac{\sinh[\pi(\omega_0 + p'_x)/k_0]}{\sinh(2\pi\omega_0/k_0)}. \quad (3.3)$$

We note that this case is not analyzed in [24], the only case when $|qE| \rightarrow 0$ is considered there. Under the same condition, the differential mean numbers of created scalar bosons follow from eq. (2.18), they are

$$N_{\mathbf{p}} \simeq e^{-\frac{\pi}{k_0}(\omega_0 - p'_x)} \frac{\cosh[\pi(\omega_0 + p'_x)/k_0]}{\sinh(2\pi\omega_0/k_0)}. \quad (3.4)$$

Note that if the kinetic energy of final particles is big enough $\frac{|qE|}{k_0\omega_0} \ll 1$, the problem can be considered perturbatively. In this case the weak time-dependent external field violates the vacuum very small, and the corresponding pair creation can be neglected in comparison with the main contribution given by Eqs. (3.3) and (3.4), which is formed in the momentum range (3.2).

The difference in distributions (3.3) and (3.4) that is stipulated by the statistics is maximal for the fast varying field when $\omega_0/k_0 \ll 1$. Then

$$N_{\mathbf{p},\sigma} \simeq \frac{1}{2} \left(1 + \frac{p'_x}{\omega_0}\right), \quad N_{\mathbf{p}} \simeq \frac{k_0}{2\pi\omega_0}. \quad (3.5)$$

In the spinless case, the mean numbers $N_{\mathbf{p}}$ given by Eq. (3.5) are unlimited growing. This is an indication of a big backreaction effect. Thus, we can suppose that for scalar QED the

concept of the external field is limited by the condition $2\pi m/k_0 \gtrsim 1$. At the same time, in the case of spinor QED, the mean number $N_{\mathbf{p},\sigma}$, given by Eq. (3.5), are limited $N_{\mathbf{p},\sigma} \leq 1$. This allows us to study fermion creation for all possible parameters given by Eq. (3.2), using the external field concept.

It follows from Eqs. (3.3) and (3.4) that for large negative longitudinal momentum,

$$p'_x < 0, |p'_x|/k_0 > K_x, \quad (3.6)$$

where $K_x \gg 1$ is a given number, the mean number of created boson and fermion pairs is exponentially small.

In what follows we show that the main contribution to the total number of created fermions is due to the sufficiently large positive longitudinal momenta p'_x from the range

$$p'_x/k_0 > K_x, \quad (3.7)$$

where it is assumed that $K_f \gtrsim K_x$. In this range, it follows from Eqs. (3.3) and (3.4) that

$$N_{\mathbf{p},\sigma} = N_{\mathbf{p}}^{\text{as}}, \quad N_{\mathbf{p}}^{\text{as}} \simeq e^{-\frac{2\pi}{k_0}(\omega_0 - p'_x)} \quad (3.8)$$

both for fermions and bosons, taking into account that for bosons $N_{\mathbf{p}} = N_{\mathbf{p}}^{\text{as}}$. We see that $N_{\mathbf{p}}^{\text{as}} \leq 1$. Note that eq. (3.8) holds true for any transversal energy $\sqrt{m^2 + \mathbf{p}_\perp^2}$. In particular, if $(p'_x)^2 \gg m^2 + \mathbf{p}_\perp^2$, distribution (3.8) can be approximated as

$$N_{\mathbf{p}}^{\text{as}} \simeq \exp\left(-\pi \frac{m^2 + \mathbf{p}_\perp^2}{k_0 p'_x}\right), \quad (3.9)$$

such that $N_{\mathbf{p}}^{\text{as}} \rightarrow 1$ as $p'_x/k_0 \rightarrow \infty$. If $(p'_x)^2 \lesssim m^2 + \mathbf{p}_\perp^2$ then distribution (3.8) can be approximated as

$$N_{\mathbf{p}}^{\text{as}} \lesssim \exp\left[-\frac{2\pi}{k_0}(\sqrt{2}-1)\sqrt{m^2 + \mathbf{p}_\perp^2}\right]. \quad (3.10)$$

We see that this expression is exponentially small in momentum range $\sqrt{m^2 + \mathbf{p}_\perp^2}/k_0 \gtrsim p'_x/k_0 > K_x$.

The above analysis shows that maximum contribution to the differential number of created fermions is provided by large positive longitudinal momenta p'_x , given by expression (3.9), with relatively small transversal momentum $|\mathbf{p}_\perp|$. Thus, taking the inequality (3.2) into account, we can conclude that the essential contribution to the total number of created fermions is due to the longitudinal momenta p'_x from the wide uniform range

$$K_x < p'_x/k_0 < \frac{|qE|}{k_0^2} - K_f, \quad (3.11)$$

where

$$|qE|/k_0^2 \gg K_x, \quad p'_x \gg m. \quad (3.12)$$

It should be noted that the contribution to the total number of created particles from the relatively narrow momentum range of the width K_x is finite and of the order K_x if $N_{\mathbf{p}} \lesssim 1$. For example, we can use this estimation for the total number of created fermions in the finite range of p'_x that is restricted by the inequality

$$-K_x < p'_x/k_0 < K_x. \quad (3.13)$$

This contribution is much less than the contribution from a very wide range (3.11). The same is true for bosons when $2\pi m/k_0 \gtrsim 1$. That is why the contribution to the total number of created fermions in the range (3.11) is the main contribution. The main contribution to the total number of created bosons is due to the range (3.11) for the slowly decaying electric field when $2\pi m/k_0 \gtrsim 1$.

Note that if $(\omega_0 - p'_x)/k_0 \gg 1$, WKB approximation holds true for $N_{\mathbf{p},\sigma}$ given by Eq. (3.8). In this domain expression (3.8) coincides exactly with an estimation, obtained previously in [23] using the semiclassical consideration, while our approximation (3.8) is valid for any value of $(\omega_0 - p'_x)/k_0$ and our exact results, given by Eqs. (3.3) and (3.4), are quite different from the semiclassical ones.

2. Total quantities

The obtained distribution $N_n = |g(-|+)|^2$ plays the role of a cut-off factor in the integral over momenta (2.20) for the total number of created particles (there for bosons $J_{(d)} = 1$). However, for bosons, this result is valid only if the electric field decays slowly enough, $2\pi m/k_0 \gtrsim 1$. Then the total number of created particles can be represented by its main contribution in the range (3.11) as follows

$$N \approx \frac{V J_{(d)}}{(2\pi)^{d-1}} \int_{p_x^{\min}}^{p_x^{\max}} dp'_x N_{p_x}^{\text{as}}, \quad N_{p_x}^{\text{as}} = \int d\mathbf{p}_{\perp} N_{\mathbf{p}}^{\text{as}}, \quad (3.14)$$

where $N_{\mathbf{p}}^{\text{as}}$ is given by eq. (3.9) and

$$p_x^{\max} = \frac{|qE|}{k_0} - K_f k_0, \quad p_x^{\min} = K_x k_0. \quad (3.15)$$

Integrating over \mathbf{p}_\perp and taking into account that $p'_x \gg m$, we find that the total number of created particles with a given longitudinal momentum reads

$$N_{p_x}^{\text{as}} \approx (k_0 p'_x)^{d/2-1} \exp\left(-\frac{\pi m^2}{k_0 p'_x}\right). \quad (3.16)$$

Using Eq. (3.16), we represent the integral (3.14) in the form

$$N \approx \frac{V J_{(d)}}{(2\pi)^{d-1}} (k_0)^{d/2-1} Y^{(1)}, \quad (3.17)$$

where $Y^{(1)}$ is the particular case of the integral

$$Y^{(k)} = \int_{p_x^{\min}}^{p_x^{\max}} dp'_x (p'_x)^{d/2-1} \exp\left(-k \frac{\pi m^2}{k_0 p'_x}\right), \quad k = 1, 2, \dots \quad (3.18)$$

Taking into account that $|qE|/k_0^2 \gg K_f \gtrsim K_x$, we obtain that the integral (3.18) is independent on the given numbers K_f and K_x in the leading order term approximation. If $m \neq 0$ then the integral (3.18) in this approximation can be expressed via the incomplete gamma function as

$$Y^{(k)} \approx \left(\frac{k_0}{\pi m^2 k}\right)^{d/2} \Gamma\left(-\frac{d}{2}, k \frac{\pi m^2}{|qE|}\right), \quad k = 1, 2, \dots \quad (3.19)$$

Note that the representation (3.19) is suitable when the electric field is weak enough, $k\pi m^2/|qE| \gg 1$. In this case one can use the following asymptotics of the incomplete gamma function,

$$\Gamma\left(-\frac{d}{2}, k \frac{\pi m^2}{|qE|}\right) \approx \exp\left(-k \frac{\pi m^2}{|qE|}\right) \left(k \frac{\pi m^2}{|qE|}\right)^{-d/2-1}. \quad (3.20)$$

For the case of a strong field, when $k\pi m^2/|qE| \ll 1$, where the case of massless fermions is included too, we find in the leading order term approximation that

$$Y^{(k)} \approx \frac{2}{d} \left(\frac{|qE|}{k_0}\right)^{d/2}. \quad (3.21)$$

Then the total number of particles created from vacuum is

$$N^{\text{strong}} \approx \frac{V J_{(d)}}{(2\pi)^{d-1}} \frac{2(|qE|)^{d/2}}{k_0 d}. \quad (3.22)$$

Finally taking into account the above results, we can represent the probability of a vacuum to remain a vacuum, defined by Eq.(A11), as

$$P_v \approx \exp\left\{-\frac{V J_{(d)}}{(2\pi)^{d-1}} \sum_{k=0}^{\infty} \frac{(-1)^{(1-\kappa)k/2}}{(k+1)^{d/2}} (k_0)^{d/2-1} Y^{(k+1)}\right\}, \quad (3.23)$$

where $Y^{(k+1)}$ is given by the integral (3.18) and can be represented in the leading term approximation with the help of Eqs. (3.19) and (3.21), respectively. For the strong field case we find that the probability P_v is determined by the total number of created particles

$$P_v^{\text{strong}} = \exp \{ -\mu N^{\text{strong}} \}, \quad \mu = - \sum_{k=0}^{\infty} \frac{(-1)^{(1-\kappa)k/2}}{(k+1)^{d/2}}. \quad (3.24)$$

One can see that dependence of the total number of particles created from vacuum by the strong exponential field on the field magnitude and space-time dimensions mimics the case of particle creation by strong T -constant electric field E (see [18]) for big T and with the identification $T = 2(k_0 d)^{-1}$. It is due to the effect of saturation for the distribution $N_n \rightarrow 1$ in the wide uniform range of initial longitudinal momentum, where there is a big increment of the kinetic momentum, $|qE|/k_0$, and $|qE|T$, for both cases, respectively.

Let us consider two strong T -exponential electric fields of the same magnitude E but with distinct parameters $k_0^{(I)}$ and $k_0^{(II)} \ll k_0^{(I)}$. Let them create from a vacuum the total numbers of particles $N^{(I)}$ and $N^{(II)}$, respectively. One can see from Eq. (3.22) that $N^{(II)} \gg N^{(I)}$, that is, the electric field of more long effective duration creates much more pairs. The total number of out-particles created from in-vacuum due to a decreasing exponential field is the same with the total number of particles created from a vacuum due to a increasing exponential field provided that the modulus of potential difference is the same for both cases. That is, we can consider $N^{(I)}$ as the total number of particles created from a vacuum due to the increasing field. We see that if $k_0^{(II)} \ll k_0^{(I)}$, the main contribution to particle creation by external electric field that switches on fast enough and then slowly decreases is due to its decreasing part, whereas the contribution from the increasing part of the field is relatively small. In particular, the exponentially decaying electric field can be treated as one, which is created by an external current that switches on fast enough and then is slowly switching off because of some dissipation processes. Thus we see that the exponentially decaying electric field under consideration allows one to study the role of switching on and switching off processes.

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Appendix A: Pair creation in a homogeneous electric field

Following general consideration in [18], we recall in this Appendix some basic elements of the generalized Furry representation [3, 4, 7] that is used to describe vacuum instability in a strong external time-dependent electric field.

For the particular case of a homogeneous electric field, we assume that the potential $A_1(t)$ ($A_\mu(t) = 0$, $\mu \neq 1$), is constant for $t < t_1$ and for $t > t_2$. Therefore, the initial (at $t < t_1$) and the final (at $t > t_2$) vacua are vacuum states of in- and out- free particles which correspond to the constant effective potentials $A_1(t_1)$ and $A_1(t_2)$, respectively. During the time interval $t_2 - t_1 = T$, the quantum Dirac field interacts with the time-dependent effective potential $A_1(t)$. In the general case, the initial and final vacua are different. We introduce an initial set of creation and annihilation operators $a_n^\dagger(\text{in})$, $a_n(\text{in})$ of in-particles (electrons), and operators $b_n^\dagger(\text{in})$, $b_n(\text{in})$ of in-antiparticles (positrons), the corresponding in-vacuum, being $|0, \text{in}\rangle$, and a final set of creation and annihilation operators $a_n^\dagger(\text{out})$, $a_n(\text{out})$ of out-electrons and operators $b_n^\dagger(\text{out})$, $b_n(\text{out})$ of out-positrons, the corresponding out-vacuum, being $|0, \text{out}\rangle$. Thus for any quantum number n , we have

$$\begin{aligned} a_n(\text{in})|0, \text{in}\rangle &= b_n(\text{in})|0, \text{in}\rangle = 0, \\ a_n(\text{out})|0, \text{out}\rangle &= b_n(\text{out})|0, \text{out}\rangle = 0. \end{aligned} \tag{A1}$$

In both cases, by $n = (\mathbf{p}, \sigma)$ we denote complete sets of quantum numbers that describe both in- and out- particles and antiparticles. The in-operators and the out-operators obey the canonical anticommutation relations. The above in- and out-operators are defined by two decompositions of the quantum Dirac field $\Psi(x)$ in the exact solutions of the Dirac equation,

$$\begin{aligned} \Psi(x) &= \sum_n [a_n(\text{in}) \psi_n(x) + b_n^\dagger(\text{in}) \bar{\psi}_n(x)] \\ &= \sum_n [a_n(\text{out}) \psi_n(x) + b_n^\dagger(\text{out}) \bar{\psi}_n(x)] . \end{aligned} \tag{A2}$$

Thus, the in-operators are associated with a complete orthonormal set of solutions $\{\zeta\psi_n(x)\}$ (we call it the in-set) of Eq. (2.1), where $\zeta = +$ stays for electrons and $\zeta = -$ for positrons. Their asymptotics at $t < t_1$ are wave functions of free particles in the presence of a constant electric potential $A_1(t_1)$. The out-operators are associated with another complete orthonormal out-set of solutions $\{\zeta\psi_n(x)\}$ of Eq. (2.1). Their asymptotics at $t > t_2$ are wave functions of free particles in the presence of a constant electric potential $A_1(t_2)$.

The inner product between two solutions $\psi(x)$ and $\psi'(x)$ of the Dirac equation on t -const hyperplane,

$$(\psi, \psi') = \int \psi^\dagger(x) \psi'(x) d\mathbf{x}, \quad (\text{A3})$$

is time-independent. Then, taking into account the structure (2.7) and initial or final forms of the functions $\zeta\psi_n(x)$ and $\zeta\psi_n$, respectively, one finds the orthonormality relations:

$$(\zeta\psi_n, \zeta'\psi_{n'}) = \delta_{n,n'}\delta_{\zeta,\zeta'}, \quad (\zeta\psi_n, \zeta'\psi_{n'}) = \delta_{nn'}\delta_{\zeta,\zeta'}. \quad (\text{A4})$$

Here we apply the standard QFT volume regularization assuming that all the processes are confined in a big D dimensional space box with the volume V . In- and out-solutions with given quantum numbers n are related by linear transformations of the form

$$\begin{aligned} \zeta\psi_n(x) &= g(+|\zeta) \psi_n(x) + g(-|\zeta) \bar{\psi}_n(x), \\ \zeta\psi_n(x) &= g(+|\zeta) \psi_n(x) + g(-|\zeta) \bar{\psi}_n(x), \end{aligned} \quad (\text{A5})$$

where the coefficients g are defined via the inner products of these sets,

$$(\zeta\psi_n, \zeta'\psi_{n'}) = \delta_{n,n'}g(\zeta|\zeta'), \quad g(\zeta|\zeta) = g(\zeta|\zeta')^*. \quad (\text{A6})$$

These coefficients satisfy the unitarity relations, which follow from the orthonormality relations (A4), and can be expressed in terms of two of them, e.g., of $g(+|+)$ and $g(-|+)$. However, even these coefficients are not completely independent,

$$|g(-|+)|^2 + |g(+|+)|^2 = 1. \quad (\text{A7})$$

A linear canonical transformation (Bogolyubov transformation) between in- and out-operators which can be derived from Eq. (A2) has the form

$$\begin{aligned} a_n(\text{out}) &= g(+|_+) a_n(\text{in}) + g(+|_-) b_n^\dagger(\text{in}), \\ b_n^\dagger(\text{out}) &= g(-|_+) a_n(\text{in}) + g(-|_-) b_n^\dagger(\text{in}). \end{aligned} \quad (\text{A8})$$

Then one can see that all the information about electron-positron creation, annihilation, and scattering in an external field can be extracted from the coefficients $g(\zeta|\zeta')$.

One of the most important quantity for the study of particle creation is differential mean number of created particles, defined as the expectation value of out number operator with respect to the in-vacuum,

$$N_n = \langle 0, \text{in} | a_n^\dagger(\text{out}) a_n(\text{out}) | 0, \text{in} \rangle = |g(-|+)\rangle^2. \quad (\text{A9})$$

It is equal to the mean number of particle-antiparticle pairs created. The total number of created particles is obtained by the summation over the quantum numbers n ,

$$N = \sum_n N_n. \quad (\text{A10})$$

The probability of a vacuum to remain a vacuum, is defined as

$$P_v = |\langle 0, \text{out} | 0, \text{in} \rangle|^2 = \exp \left\{ \kappa \sum_n \ln(1 - \kappa N_n) \right\}, \quad (\text{A11})$$

where $\kappa = +1$ for fermions and $\kappa = -1$ for bosons. The probability of the electron scattering $P(+|+)_{n,n'}$ and the probability of a pair creation $P(-+|0)_{n,n'}$ are, respectively

$$\begin{aligned} P(+|+)_{n,n'} &= |\langle 0, \text{out} | a_n(\text{out}) a_{n'}^\dagger(\text{in}) | 0, \text{in} \rangle|^2 = \delta_{n,n'} \frac{1}{1 - \kappa N_n} P_v, \\ P(-+|0)_{n,n'} &= |\langle 0, \text{out} | b_n(\text{out}) a_{n'}(\text{out}) | 0, \text{in} \rangle|^2 = \delta_{n,n'} \frac{N_n}{1 - \kappa N_n} P_v. \end{aligned} \quad (\text{A12})$$

The probabilities for a positron scattering and a pair annihilation are given by the same expressions $P(+|+)$ and $P(-+|0)$, respectively.

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